GETTING A BETTER ANGLE

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Since *Consortium* now contains lots of interesting material on modeling, it is time to close the Modeler's Corner and start a new column. As some of you may have noticed, many of my columns have had a distinctive geometric flavor so it should come as no surprise that the new column will be called Geometer's Corner. The goal of the column is to provide all of you who teach, geometry material that deals with topics that are not traditionally taught but that may have a place in the geometry classroom of the 21st century. I welcome any and all suggestions for future columns. I can be reached at jchoate@groton.org.

This month's column deals with one of my favorite problems. I originally found it in Heinrich Dorrie's *100 Great Problems in Elementary Mathematics* [1], a wonderful book now out in paperback that I highly recommend. The problem is often referred to as the Problem of Regiomontanus and is reputed to be the first max-min problem of the modern age.

> "At what point on the ground does a perpendicularly suspended rod appear largest [i.e., subtends the greatest visual angle]? It has been claimed that this was the first extreme problem in the history of mathematics since antiquity."[i]

A version of this problem is found in many calculus textbooks. The following version comes from Anton's *Calculus* [2].

> The lower edge of a painting, 10 ft. in height, is 2 feet above an observer's eye level. Assuming that the best view is obtained when the angle subtended at the observer's eye by the painting is maximum, how far from the wall should the observer stand?[ii]

I use a problem similar to this in my geometry class. I first introduce it when we begin to study right angle trig. All my students have TI-89 graphing calculators and they have learned how to use the inverse trig functions to find angles. To start we assume that the painting is 6 feet tall and is hanging in such a way that the bottom of the picture is 4 feet above an observer's eye level, which is 5 feet above the ground.

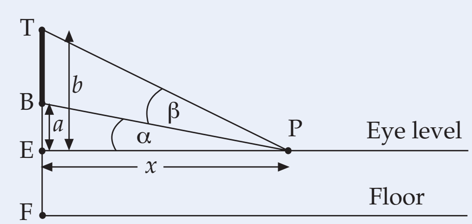


Figure 1. Regiomontanus's Viewing Angle Problem.

Figure 1 illustrates the problem. The hanging picture is represented by segment TB, and the viewing angle is represented by ∠BPT. PE represents the distance from the wall. At this point, you can have students calculate ∠BPT = ∠EPT – ∠EPB for different values of PE by using the formulas

$$\angle EPT = \tan^{-1}\left(\frac{10}{PE}\right) \text{ and } \angle EPB = \tan^{-1}\frac{4}{PE}.$$

If they put their results in table form they should get a table that looks like the one shown in **Table 1.**

PE	$\tan^{-1}\left(\frac{10}{PE}\right)$	$\tan^{-1}\frac{4}{PE}$	∠BPT
1	84.2895	75.9638	8.3257
2	78.6901	63.4350	15.2551
3	73.3008	53.1301	20.1707
4	68.1986	45.0000	23.1986
5	63.4350	38.6598	24.7752
6	59.0363	33.6901	25.3462
7	55.0080	29.7449	25.2631
8	51.3402	26.5651	24.7752
9	48.0128	23.9625	24.0503
10	41.9872	21.8014	20.1858

TABLE 1. VALUES FOR ∠BPT.

Once they have filled in the table have them plot the graph of ∠BPT in terms of PE. The graph gives some interesting information about the problem.

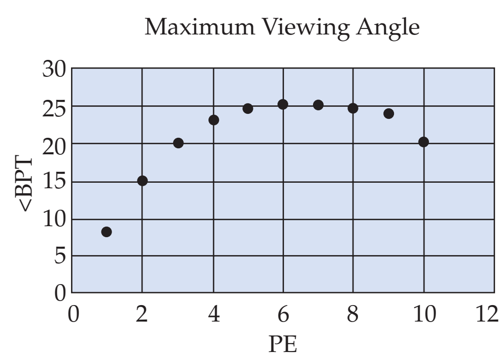


FIGURE 2. THE GRAPH OF ∠BPT AS A FUNCTION OF PE.

Hopefully they will see that the graph shows that there might be a maximum value for ∠BPT. Have them find this value to the nearest 1/100th using their calculators and a guess-and-check method of their choosing.

If you have access to a geometric construction program such as Cabri or Geometer's Sketchpad, you could use it to find another approximation by creating a sketch like the one in Figure 1 with P being a movable point. Once you have an approximation, ask students to convince themselves that for any value of the viewing angle less then the maximum there are two places where the viewer could stand and achieve that angle. They can argue this both by using the graph they created earlier or by using the sketch they created with the movable point P. For example, in Table 1 it shows that ∠BPT = 24.57772° for EP = 5 and EP = 8. At this point, they haven't found the exact solution but they do know that (a) one exists and (b) for any value less than the maximum there are two places the viewer can stand.

There are two non-calculus ways to find the exact solution, depending on what course you are teaching. In a geometry course I'd return to the problem when you begin a study of circles. Let's come back to the circles solution later. In a trig course you can continue with an algebraic solution such as the one given in Eli Maor's wonderful book *Trigonometric Delights* [3], which, thanks to the generosity of the good people at the Princeton Press, is available for free on the Internet. Here is Maor's solution[iii], which makes use of the fact that the arithmetic mean of two numbers is always greater than or equal to the geometric mean. This means that for all $u, v, >0$, $\frac{u+v}{2} \geq \sqrt{uv}$. Maor notes that to find the max value for the viewing angle ∠BPT one can look for the max value of tan(∠BPT) or the minimum value of cot(∠BPT). He wisely chooses for algebraic reasons to go with minimizing cot(∠BPT).

Here is his solution.

In what follows b = TE, a = BE, β = ∠EPT, α = ∠EPB, θ = ∠BPT = $\beta - \alpha$, and x = PE (see Figure 1).

Now, $\cot(\beta) = \frac{x}{b}$, $\cot(\alpha) = \frac{x}{a}$, and $\cot(\theta) = \cot(\beta - \alpha)$.

Using the formula for $\cot(\beta - \alpha)$ and making some substitutions, you get

$$\cot(\beta - \alpha) = \frac{\cot(\alpha)\cot(\beta)+1}{\cot(\alpha)-\cot(\beta)}$$

$$= \frac{\left(\frac{x}{a}\right)\left(\frac{x}{b}\right)+1}{\frac{x}{a}-\frac{x}{b}}$$

$$= \frac{x}{b-a} + \frac{ab}{(b-a)x}$$

Here is where the relationship between the arithmetic and geometric mean of two numbers comes in.

Let $u = \frac{x}{b-a}$ and $v = \frac{ab}{(b-a)x}$. Since $\frac{u+v}{2} \geq \sqrt{uv}$, we have

$\frac{x}{b-a} + \frac{ab}{(b-a)x} \geq 2\sqrt{\left(\frac{x}{b-a}\right)\left(\frac{ab}{(b-a)x}\right)} = 2\frac{\sqrt{ab}}{b-a}$.

There is equality when $u = v$ or when $\frac{x}{b-a} = \frac{ab}{(b-a)x}$. This implies that $x^2 = ab$ and finally that $x = \sqrt{ab}$.

So the exact solution to our original problem is that the person would have to stand $\sqrt{40}$ feet to get the maximum viewing angle. Neat—no calculus—just some trig and some clever algebra.

If you are introducing the problem in a geometry class, here is a geometric solution that requires some knowledge about inscribed angles in a circle and intersecting secant lines. Earlier, we saw that given an angle less than the maximum angle one could always find two points, call them P_1 and P_2 such that $\angle BP_1T = \angle BP_2T$. This is illustrated in **Figure 3.**

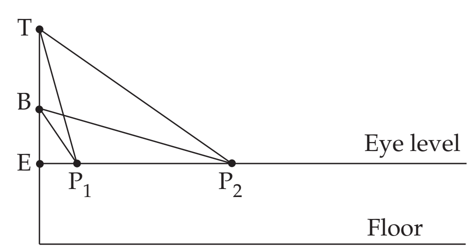


FIGURE 3. $\angle BP_1T = \angle BP_2T$

There is something special about points B, T, P_1 and P_2 that will jump out at you if you think of the segment BT as being a chord of a circle and $\angle BP_1T$ and $\angle BP_2T$ as being inscribed angles in a circle. All four points lie on a circle! If P is the point where the maximum occurs, then the circle through B and T will intercept the eye level line in only one place and hence it must be tangent at that point. Now the picture looks like **Figure 4.**

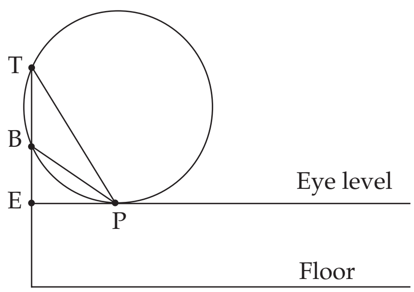


FIGURE 4. THE EXACT SOLUTION TO THE VIEWING ANGLE PROBLEM.

There is a theorem that relates secant lines to tangent lines, which says in this case that $TE \cdot BE = EP^2$. In our example, TE = 10, BE = 4 so EP = $\sqrt{40}$, the same answer we got analytically.

Let's go a bit farther with the problem and come up with a way to construct the circle tangent to the eye level line using only Euclidian tools. Here is one way of doing it.

Locate a point Q on line BE such that QE = BE and Q-E-B.

Construct the midpoint M of segment TQ.

Construct the circle with center M that passes through point T.

Construct a perpendicular to TQ through E and label its intersections with circle M as S_1 and S_2.

S_1 is the desired point of tangency.

Construct the circle C_2 through T, B and S_1.

Note that ΔQS_1T is a right triangle and S_1E is the altitude to its hypotenuse. Since the altitude to the hypotenuse is the geometric mean of the two segments into which it divides the hypotenuse you can show that $S_1E^2 = QE \cdot TE$. S_1 is the desired point of tangency.

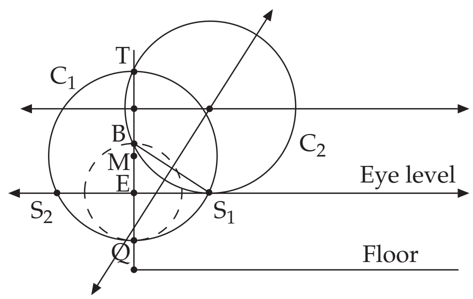


FIGURE 5. GEOMETRIC CONSTRUCTION OF MAXIMUM VIEWING ANGLE.

An interesting variation to the Art Gallery Problem is finding the best place to sit in a movie theatre with raked seating. I found the following version on the Grand Valley State University Mathematics Department Website located at www.gvsu.edu/math/calculus/M201/pdf/movie.pdf.

A movie theatre has a screen that is positioned at 10 feet off the floor and is 25 feet high. The first row of seats is placed 9 feet from the screen and the rows are 3 feet apart. The floor of the seating area is inclined at an angle of 20 degrees above the horizontal. Suppose your eyes are 4 feet above the floor and you want to locate a seat that gives you the maximum viewing angle. How far up the inclined floor should you locate the seat?[iv]

In the **Figure 6,** TB = 25, BG = 10, GA = 9, $\angle DAE = 20°$ and CA = 4 and line PH is parallel to line AD. This problem has a similar constructive solution to that of the art gallery. You need to find a point P such that the circle through T, B, and P is tangent to line PH. In order to do this, you need to construct the point H, which is where the line that determines your eye level intersects the wall. Once you have that point located you need to find the length of HP, the geometric mean of TH and BH. The length of segment CP is how far you should place your seat up the raked floor.

Using the law of cosines, you can find an expression for $\angle BPT$. Start with a coordinate system with origin at G. In this system T = (0, 35), B = (0, 10), and A = (9, 0). Since line AD has slope tan(20°) and goes through the point (9, 0), its equation in point-slope form is $y = \tan(20°)(x - 9)$. Line PH is parallel to line AD and is 4 units above it so its equation is $y - 4 = \tan(20°)(x - 9)$. Therefore, any point on line PH has coordinates $(x, \tan(20°)x - 9\tan(20°) + 4)$.

Expressing BP and TP in terms of x gives you

$$BP = \sqrt{x^2 + \left(10 - \left(\tan\left(20^\circ\right)x - 9\tan\left(20^\circ\right) + 4\right)\right)^2}$$

$$TP = \sqrt{x^2 + \left(35 - \left(\tan\left(20^\circ\right)x - 9\tan\left(20^\circ\right) + 4\right)\right)^2}$$

$\angle BPT$ can now be found using the Law of Cosines.

$$\angle BPT = \cos^{-1}\left(\frac{BT^2 + TP^2 - 625}{2BT \cdot PT}\right)$$

A maximum value for this function can now be found using a graphing calculator with a maximum function or by calculus. Once you have the

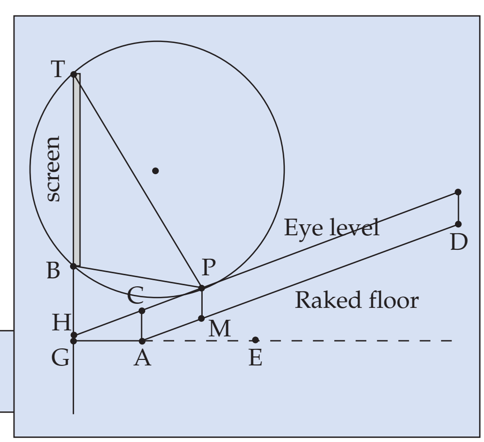


FIGURE 6. MAXIMUM VIEWING ANGLE IN A MOVIE THEATRE WITH RAKED SEATING.

coordinates of P, the length AM can be found, and to the nearest hundredth it is 8.25. Another solution using calculus can be found at www.gvsu.edu/math/calculus/M201/pdf/movie.pdf. This solution is derived from UMAP Module 729, "Calculus in a Movie Theatre."

It is interesting that both variations of Regiomontanus's problem can be solved exactly by construction. Both require you to do the following:

Given two points A and B and a line that does not contain A or B and that does not intersect segment AB, construct the circle that passes through A and B which is tangent to the given line.

I leave it to the interested reader to develop a Cabri or Geometer's Sketchpad solution to the movie theatre problem using a construction similar to the one used to find the solution to the Art Gallery problem.

Regiomontanus's original problem about the suspended rod can also be solved using the following construction:

Given two points A and B and a circle with center C such that A and B are external to C; A, B and C are collinear and A-B-C, construct the circle through A and B that is tangent to circle C.

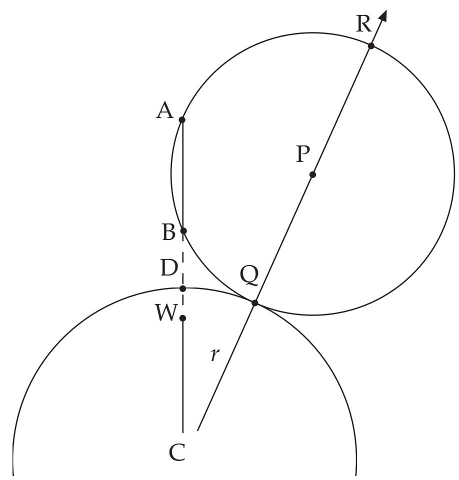
Here is a construction using Geometer's SketchPad. In **Figure 7,** circle P is tangent to circle C at Q. Q, C, and P are collinear, and CP extended meets circle P at R. Since secants AB and QR intersect at C, AC · BC = RC · QC. AC, BC, and CQ are known. If we let CQ = r and QP = s, then RQ = 2s, and RC = 2s + r. Therefore,

$$AC \cdot BC = (2s + r)r$$

$$= 2sr + r^2$$

and

$$s = \frac{AC \cdot BC - r^2}{2r}$$



AB = 12.2771 cm
BC = 8.3759 cm
r = 6.2475 cm

$$\frac{AB \cdot BC - r^2}{2r} = 5.1061 \text{ cm}$$

$$\frac{\frac{AB \cdot BC - r^2}{2r}}{r} = 0.82$$

FIGURE 7. SOLUTION OF REGIOMOTANUS'S ORIGINAL PROBLEM.

Here is how to do the construction shown in **Figure 7.**

Step 1. Measure AC, BC, and CD = r, the radius of the given circle.

Step 2. Use the SketchPad's calculator to calculate $s = \frac{AC \cdot BC - r^2}{2r}$

Step 3. With C as center dilate point D by a scale factor of s/r, creating a point W. Construct segment CW. Note that CW has length s.

Step 4. With A as center and CW as radius construct circle A. With B as center and CW as radius construct circle B. Label one of the points of intersection of the two circles P.

Step 5. Construct a circle with P as center that contains point A. This is the circle tangent to circle C.

Step 6. Label the intersection of circle P with circle C, Q.

∠AZB is the maximum angle!

If you would like a challenge, try the following.

Given a circle C and a line l that does not intersect C, let A and B be any two points on l. Find the point P on circle C such that ∠APB is a maximum.

If you come up with a solution please send it to me and I'll publish it in the next Geometer's Corner. ❏

References

[1] Dorrie, H, *100 Great Problems in Elementary Mathematics*, Dover, 1965, ISBN 486-61348-8

[2] Anton, Bivens, Davis, *Calculus, 7th Edition*, Wiley, 2003, ISBN 0-471-38157-8

[3] Maor, Eli, *Trigonemteric Delights*, Princeton University Press, 2002, ISBN 0691095418

[i] Dorrie, page 369

[ii] Anton, page 499

[iii] Maor, page 46

[iv] GVSU web site, page 1

Send solutions to old problems and any new ideas to the Geometer's Corner editor: Jonathan Choate, Groton School, Box 991, Groton, MA 01450.